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# ENERGY BALANCE AND THE SPEED OF CRACK GROWTH IN A COMPRESSED PLATE WITH DELAMINATION

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**Abstract**—The evolution of the crack growth speed in a buckled one-dimensional delamination model is studied and two approximate solutions are presented. In the quasi-dynamic analysis one assumes that the time-dependent deflection of the delaminated layer may be approximated by the static postbuckling solution for the current delamination length. In a refined analysis one introduces an indeterminate amplitude function. The local growth condition at the crack tip is not enforced but a global energy-balance condition is used. The crack growth speeds are found to be comparable to the speeds of flexural waves. For slow and moderate rates of crack growth, the present results are in close agreement with the finite-difference solutions of the dynamic problem.

## NOMENCLATURE

| | |
|---|---|
| $h$ | delamination thickness |
| $2a_0$ | initial length of delamination |
| $2a(t)$ | current length of delamination |
| $\xi(t) =$ | $a(t)/a_0$ |
| $E, \nu$ | elastic moduli |
| $\rho h$ | mass per unit length of the delaminated layer |
| $w(x, t)$ | deflection of the layer |
| $P(t)$ | axial force in the layer |
| $\varepsilon_0$ | axial strain in the base plate |
| $G$ | strain energy release rate |
| $G_0^*$ | specific fracture energy for the initiation of growth |
| $G^*$ | dynamic specific fracture energy |
| $\alpha =$ | $12\varepsilon_0(a_0/\pi h)^2$ |
| $\gamma^* =$ | $\dfrac{24(1-\nu^2)}{Eh} G^*(a_0/\pi h)^4$ |
| $c_0 =$ | $\dfrac{\pi h}{2a_0}\left(\dfrac{E}{12(1-\nu^2)\rho}\right)^{1/2}$ |
| $V(\xi) =$ | $\dot{a}(t)/(\pi c_0)$ |
| $A(\xi)$ | normalized amplitude of the improved solution [eqn (26)]. |

## 1. INTRODUCTION

Under a sufficiently large compression load, a composite laminate with an interior delamination may be susceptible to the initiation of local buckling and to postbuckling delamination growth. Some understanding of the effects of the various geometrical, material and loading parameters, and of the ways in which they combine to produce qualitatively different types of buckling, postbuckling and delamination growth behavior, has emerged recently as a result of several analytical studies on the subject [for selected references see Storakers (1989) and Yin (1989)]. These studies were based on static postbuckling solutions of one- and two-dimensional delamination models, and assumed the Griffith criterion of a constant specific fracture energy in the growth process. The nature of the buckling and postbuckling behavior was shown to be crucially dependent on the relative slenderness of the delaminated layer versus the base laminate (Yin and Fei, 1984, 1988; Simitses *et al.*, 1985; Yin *et al.*, 1986). Significant mechanical interaction between the layer and the base laminate was noticed in the postbuckling process, and such interaction affects the continued opening or possible closing of the crack surfaces at the front of delamination. In the case of delaminated layers with unsymmetric layups, the buckling load and the postbuckling deformation may be strongly affected by bending–stretching coupling (Yin, 1986, 1988).

Although these analytical studies took into account the important effects of geometrical nonlinearity, they ignored the inertial effect, which may significantly affect the deflection and the force and moment resultants, particular in the boundary region of the delaminated layer. According to the postbuckling equilibrium solutions of strip and circular delamination models, if the strain load in the base laminate is maintained constant after the initiation of growth, then the energy-release rate at the crack front increases rapidly in an initial stage of growth. But the energy-release rate in dynamic growth is always equal to the specific fracture energy of the material, although the latter may depend on the mix of fracture modes and on the speed of crack growth. These two different predictions of the energy-release rate suggest that, at least within a small neighborhood of the moving crack boundary, the deflection of the static solution may be significantly modified by the inertial effect. Presumably, the dynamic deflection has a smaller curvature and therefore yields a smaller bending moment at the boundary of delamination. This in turn constrains the energy release rate to the level dictated by the crack growth criterion.

The free boundary problem associated with the dynamic growth of a buckled delamination cannot be solved in closed form and is computationally laborious. Consequently, it is desirable to obtain simple yet reasonably accurate estimates of the growth speed of a one-dimensional delamination from the condition of global energy balance, by making plausible assumptions concerning the approximate shape of the dynamic deflection. "Quasi-dynamic analysis" based on a similar consideration has been applied previously to related problems of crack growth in a double cantilevered beam specimen subjected to a constant load or a constant end separation (Berry, 1960a, b). For the cleavage fracture of the same specimen caused by the insertion of a moving wedge, Bilek and Burns (1974) obtained a dynamic solution by using the similarity method. The solution yields a crack growth speed close to the prediction of the quasi-dynamic analysis. This appears to provide some support for the validity of the quasi-dynamic solution (Burns and Webb, 1970).

In the present paper, the global energy-balance condition is used to investigate the evolution of the crack speed in uni-directional and bi-directional growth of a thin strip (one-dimensional) delamination. We assume that an axial compressive strain is imposed in the base laminate to cause buckling of the delaminated layer, and the load is increased further to a level sufficient for the initiation of delamination growth. The imposed axial strain is maintained constant in the subsequent growth process. Besides the quasi-dynamic solution based on the equilibrium deflection function, we obtain an improved approximate solution containing a time-dependent amplitude function which is to be determined by a weighted integral of the equation of motion.

The results of the analysis indicate that, after an initial period of accelerating growth, the crack speed levels off in delamination models with a small specific fracture energy. In models with a relatively large specific fracture energy, the growth eventually decelerates to a state of arrest. Close agreement between the growth speeds of the two solutions is obtained when the speeds are not significantly greater than the speed of the flexural waves having a wavelength equal to the initial length of the delamination.

## 2. FORMULATION

We consider the simplest problem of delamination growth, namely, dynamic growth of a thin, strip delamination in a thick homogeneous base plate. The delamination runs across the entire width of the plate, so that the geometry of the problem is independent of the $z$-coordinate (Fig. 1). The thickness of the delaminated layer is assumed to be negligibly small compared to that of the base plate so that buckling of the delaminated layer and growth of the delamination do not affect the state of membrane strain in the base plate. Such a delamination is called a "thin-film" delamination. Let $h$ and $2a_0$ denote, respectively, the thickness and the initial length of the delaminated layer. The layer buckles when the compressive axial strain $\varepsilon_0$ in the base plate exceeds the bifurcation strain $(\pi h/a_0)^2/12$. Prior

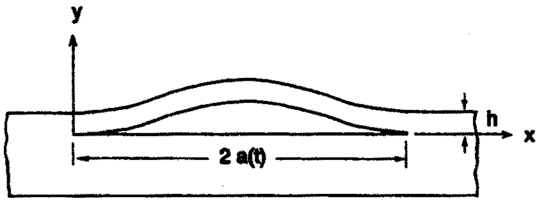


Fig. 1. One-dimensional delamination model.

to the growth of the delamination, the postbuckling deformation of the layer is described by

$$w_0(x) = 2h\sqrt{(a_0/\pi h)^2\varepsilon_0 - 1/12}[1-\cos(\pi x/a_0)], \quad (0 < x < 2a_0). \tag{1}$$

The static strain energy release rate $G$ at the front of the delamination may be determined by evaluating the J-integral around a path enclosing the crack tip (Yin and Wang, 1984). This yields

$$G = \frac{Eh}{2(1-\nu^2)}(\varepsilon_0 - \tfrac{1}{12}(\pi h/a_0)^2)(\varepsilon_0 + \tfrac{1}{4}(\pi h/a_0)^2). \tag{2}$$

Delamination growth starts when $\varepsilon_0$ is sufficiently large so that $G$ attains the fracture toughness $G_0^*$. If the axial strain is maintained constant after the initiation of growth, the deflection function $w(x, t)$ in the subsequent growth process is governed by the equations

$$Dw_{,xxxx} + P(t)w_{,xx} + \rho h w_{,tt} = 0, \quad (0 < x < 2a(t)), \tag{3}$$

$$w(x, t) = 0, \quad (x < 0 \text{ or } x > 2a(t)), \tag{4}$$

where $D = Eh^3/[12(1-\nu^2)]$ is the bending rigidity, $E$ and $\nu$ are the elastic moduli of the material, and $\rho h$ is the mass per unit length of the delaminated layer. In eqn (3), the subscripts following a comma indicate partial differentiation. We here assume that the delamination grows at the right-hand end, $x = 2a(t)$, but not at the left-hand end (the case of bi-directional growth will be considered later). It is anticipated that the speed of crack growth is comparable to the speed of flexural waves whose wavelengths are of the order of the initial length of the delamination. For thin delaminations these speed are considerably smaller than the longitudinal wave speed. Hence the axial compressive load $P$ does not depend appreciably on $x$.

The deflection function $w(x, t)$ must satisfy the following boundary conditions:

$$w = w_{,x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 2a(t), \tag{5}$$

and the initial conditions

$$w(x, 0) = w_0(x), \quad w_{,t}(x, 0) = 0, \tag{6}$$

where $w_0(x)$ is as defined by eqn (1). Furthermore, a boundary condition for the axial displacement requires that

$$P(t) = \frac{Eh}{(1-\nu^2)}\left(\varepsilon_0 - \frac{1}{4a(t)}\int_0^{2a(t)} (w_{,x})^2\,\mathrm{d}x\right). \tag{7}$$

Finally, one has the delamination growth condition at the crack tip:

$$\rho h(w_{,t})^2 + D(w_{,xx})^2 + \frac{12D}{(Eh)^2}[E\varepsilon_0 - (1-\nu^2)P(t)/h]^2 = 2G^* \quad \text{at} \quad x = 2a(t). \tag{8}$$

The preceding governing differential equation and growth condition may be derived from Hamilton's principle, as was done by Bilek and Burns (1974) for dynamic fracture of double cantilevered beams and by Bottega and Maewal (1983) for the growth of mid-plane circular delaminations under axisymmetric transient loads. It should be mentioned that the first term on the left-hand side of eqn (8) always vanishes and therefore may be ignored. To prove this assertion, we form the material time derivative of eqn (4) in the region $x > 2a(t)$ and take the limit $x \to 2a(t)$ from the right. We then obtain

$$\dot{a}(t)w_{,x} + w_{,t} = 0 \quad \text{at} \quad x = 2a(t).$$

This yields $w_{,t}(2a(t), t) = 0$ since $w_{,x}$ vanishes at the moving crack tip.

By substituting $P(t)$ of eqn (7) into eqns (3) and (8), one obtains a nonlinear integro-differential equation and a moving boundary condition for the deflection $w(x, t)$. Here the unknown function $a(t)$ appears not only in the boundary condition but also in the governing equation of motion. This feature and the nonlinearity of the equation distinguish the present problem from the usual free boundary problems, including the various dynamic fracture problems related to the cleavage of double cantilevered beam specimens (Berry, 1960a, b; Burns and Webb, 1970; Bilek and Burns, 1974) and the tearing and lifting of adhesive layers and beams from a substrate (Burridge and Keller, 1978; Hellan, 1978). Special analytical techniques such as the similarity method may not be applicable to the present problem, in spite of their success in other cases.

The specific fracture energy $G^*$ of eqn (8) stands for a critical level of the crack driving force, or the energy supply rate, required for sustaining crack growth. Although $G^*$ is generally a function of the crack growth speed and the temperature, a constant value of $G^*$ is usually assumed in the various analytical studies of dynamic fracture and this assumption will be adopted in the present analysis. However, this constant value of $G^*$ for sustaining the growth need not be equal to the corresponding value $G_0^*$ for the initiation of growth. The latter is given by the right-hand side of eqn (2), if $\varepsilon_0$ is taken to be the axial compressive strain at the initiation of growth. If we define the dimensionless constants $\gamma^*$ and $\alpha$ by the expressions

$$\gamma^* = \left(\frac{a_0}{\pi}\right)^4 \frac{2G_0^*}{h^2 D}, \quad \alpha = 12\varepsilon_0(a_0/\pi h)^2. \tag{9a, b}$$

Then, by setting $G = G_0^*$ in eqn (2), we obtain

$$\gamma^* = (\alpha - 1)(\alpha + 3)/12. \tag{10}$$

Either one of the two constants $\alpha$ and $\gamma^*$ may be used to characterize the static specific fracture energy $G_0^*$, and to determine the critical axial strain $\varepsilon_0$ at the initiation of delamination growth. These constants, however, depend not only on the material but also on the initial geometry of the delaminated layer through the factor $a_0^4/h^5$ in eqn (9a).

## 3. QUASI-DYNAMIC SOLUTION

By assuming that the static beam solutions represent the deflection at the successive instants of crack growth, the quasi-dynamic analysis allows the speed of crack growth to be obtained directly from a global energy-balance law, without recourse to the equation of motion or to the local growth condition at the crack tip. For a buckled, thin strip delamination, the deflection function of the quasi-dynamic solution is obtained by substituting $a(t)$ for $a_0$ in eqn (1). This yields

$$w(x,t) = 2h\sqrt{(a(t)/\pi h)^2\varepsilon_0 - 1/12}\left(1 - \cos\frac{\pi x}{a(t)}\right). \tag{11}$$

The problem reduces to the determination of the function $a(t)$ or its derivative. Substituting eqn (11) into eqn (7), one obtains

$$P(t) = D[\pi/a(t)]^2. \tag{12}$$

At the initiation of delamination growth, one has

$$w(x,0) = w_0(x) = h[(\alpha-1)/3]^{1/2}[1-\cos(\pi x/a_0)], \tag{13}$$

$$(1-\nu^2)P(0)/(Eh) = (1/12)(\pi h/a_0)^2 = \varepsilon_0/\alpha, \tag{14}$$

where eqn (9b) has been used. The membrane and bending energies of the delaminated layer in this state are given respectively by

$$\frac{Eh}{2(1-\nu^2)}\left(\frac{(1-\nu^2)P(0)}{Eh}\right)^2 2a_0 = \frac{Eha_0}{1-\nu^2}(\varepsilon_0/\alpha)^2$$

and

$$(D/2)\int_0^{2a_0}(w_{,xx})^2\,\mathrm{d}x = \frac{2Eha_0\varepsilon_0^2}{1-\nu^2}\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right).$$

The same layer in its unbuckled reference state has the strain energy $Eha_0(\varepsilon_0)^2/(1-\nu^2)$. Hence in the static buckling process the layer releases an amount of strain energy

$$-\Delta U_0 = \frac{Eha_0\varepsilon_0^2}{1-\nu^2}\left(1-\frac{1}{\alpha}\right)^2. \tag{15}$$

We define the normalized half-length of the delamination at time $t$ by the expression

$$\xi(t) = a(t)/a_0. \tag{16}$$

Then

$$w(x,t) = h\left(\frac{\alpha\xi^2-1}{3}\right)^{1/2}\left(1-\cos\frac{\pi x}{a_0\xi}\right), \tag{17}$$

$$P(t) = \frac{Eh\varepsilon_0}{(1-\nu^2)\alpha\xi^2}. \tag{18}$$

Steps similar to those leading to eqn (15) yield the release of strain energy in the buckling of the layer of length $2a(t)$:

$$-\Delta U = \frac{Eha_0\varepsilon_0^2}{1-\nu^2}\xi\left(1-\frac{1}{\alpha\xi^2}\right)^2. \tag{19}$$

Comparing eqns (15) and (19), one finds that the release of the strain energy in the course of the quasi-dynamic growth from the initial length $2a_0$ to the final length $2a_0\xi(t)$ is

$$\Delta U_0 - \Delta U = \frac{Eha_0\varepsilon_0^2}{1-\nu^2}\left[\xi\left(1-\frac{1}{\alpha\xi^2}\right)^2 - (1-1/\alpha)^2\right]. \tag{20}$$

The global energy-balance condition postulates that this released strain energy equals the sum of the final kinetic energy and the fracture energy consumed in the growth process. The fracture energy is given by

$$2(\xi-1)a_0G^* = (G^*/G_0^*)2a_0(\xi-1)G_0^*$$
$$= \frac{G^*}{G_0^*}\frac{Eh\varepsilon_0^2a_0}{1-\nu^2}(1-1/\alpha)(1+3/\alpha)(\xi-1). \qquad (21)$$

The kinetic energy at the final state is

$$T = (\rho h/2)\int_0^{2a(t)}(w_{,t})^2\,\mathrm{d}x,$$

where

$$w_{,t} = (\partial w/\partial a)\dot{a}(t)$$
$$= 2h\dot{a}(t)\left(\frac{\varepsilon_0 a}{(\pi h)^2}[\varepsilon_0(a/\pi h)^2-1/12]^{-1/2}(1-\cos\pi x/a)-\sqrt{\varepsilon_0(a/\pi h)^2-1/12}\,\frac{\pi x}{a^2}\sin(\pi x/a)\right).$$

It follows that

$$T = \frac{Eha_0\varepsilon_0^2}{1-\nu^2}(a/\pi c_0)^2\frac{6\xi}{\alpha}\left(1+\frac{1}{(1-1/\alpha\xi^2)}+(4\pi^2/9-1/6)(1-1/\alpha\xi^2)\right), \qquad (22)$$

where

$$c_0 = \frac{\pi}{2a_0}\left(\frac{Eh^2}{12(1-\nu^2)\rho}\right)^{1/2} \qquad (23)$$

is the speed of flexural waves in the delaminated layer with a wave length $2a_0$ (i.e. with a wave form identical to the initial buckled shape of the layer).

By equating eqn (20) to the sum of eqns (21) and (22), we obtain the delamination growth speed according to the quasi-dynamic analysis:

$$(\dot{a}/\pi c_0)^2 = \frac{\left(1-\frac{1}{\alpha\xi^2}\right)^2-\frac{1}{\xi}\left(1-\frac{1}{\alpha}\right)^2-(G^*/G_0^*)\left(1-\frac{1}{\alpha}\right)\left(1+\frac{3}{\alpha}\right)\left(1-\frac{1}{\xi}\right)}{\frac{3}{2\alpha}\left(\frac{1}{1-\frac{1}{\alpha\xi^2}}+1+\frac{1}{3}(4\pi^2/3-1/2)\left(1-\frac{1}{\alpha\xi^2}\right)\right)}. \qquad (24)$$

The acceleration of the delamination front is given by

$$\ddot{a} = \frac{\mathrm{d}\dot{a}}{\mathrm{d}\xi}\,\dot{\xi} = \frac{(\pi c_0)^2}{2a_0}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{a}{\pi c_0}\right)^2.$$

At the initial state, one has

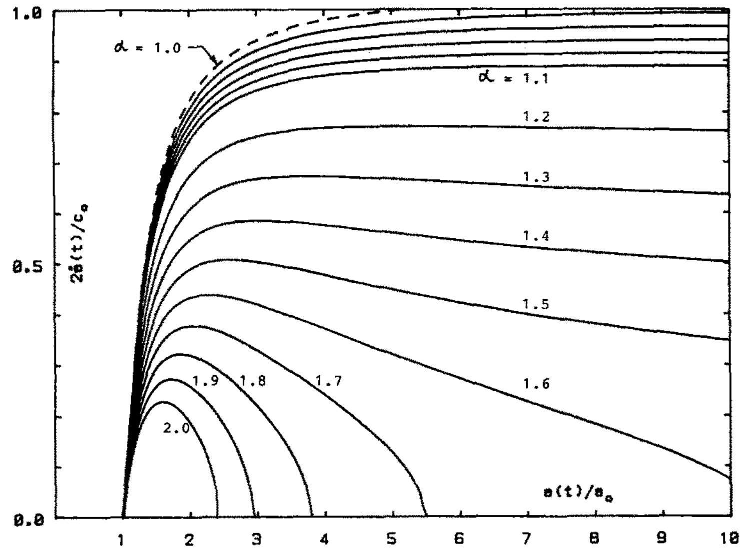


Fig. 2. Uni-directional growth speed—quasidynamic solution.

$$\ddot{a} = \left(\frac{2(\pi c_0)^2}{a_0}\right)\frac{(1-G^*/G_0^*)(1-1/\alpha)(1+3/\alpha)}{\dfrac{\alpha}{\alpha-1}+1+\frac{1}{3}(4\pi^2/3-1/2)(1-1/\alpha)}.$$

Hence the initial acceleration is positive if the dynamic specific fracture energy $G^*$ is smaller than the static value $G_0^*$. The initial acceleration vanishes if $G^* = G_0^*$. For the latter case, the change of the normalized growth speed $2\dot{a}(\pi c_0)$ with the increase of the normalized delamination length $\xi$ is shown in Fig. 2 for several values of the material parameter $\alpha$. Smaller values of $\alpha$ correspond to faster growth behavior and, as $\alpha \to 1$, the curves approach a limiting asymptote which is shown by the dashed curve in the figure. According to eqns (9a) and (10), $\alpha$ must be greater than unity and, for a specified delamination thickness $h$ and initial length $a_0$, larger values of $\alpha$ correspond to larger fracture toughness $G_0^*$. For $G^* = G_0^*$, the right-hand side of eqn (24) is positive for arbitrarily small $\xi$ only if $\alpha < 3$. If $\alpha$ is within the range $1.5 < \alpha < 3$, then the crack speed decreases to zero when the normalized length $\xi$ increases to the value $1/[\{2(\alpha-1)\}^{1/2}-1]$. On the other hand, if $1 < \alpha < 1.5$, then the delamination growth continues indefinitely with the normalized crack growth speed approaching a terminal value

$$\lim_{\xi\to\infty}\left(\frac{\dot{a}}{\pi c_0}\right) = 2\left(\frac{3/(2\alpha)-1}{5.5+4\pi^2/3}\right)^{1/2}. \tag{25}$$

It is interesting to observe that, although the driving force of the cleavage fracture of a double cantilevered beam specimen is quite different from that of the delamination buckling, qualitatively similar relations between the growth speed and the crack length are obtained in the two cases [compare figure 7 of Berry (1960b) and Fig. 2 of the present paper].

## 4. REFINED SOLUTION

A natural improvement over the quasi-dynamic analysis may be based on the assumption that the deflection function has the shape of the postbuckling equilibrium solution at each instant but differs from the latter by a factor depending on time or, equivalently, on the current length of the delamination. The amplitude function introduced by this assumption, and the time-dependence of the delamination length, will be solved from the global energy-balance condition and a weighted integral of the equation of motion.

Henceforth eqn (17) will be replaced by

$$\frac{w(x,t)}{h} = A(\xi)\left(1-\cos\frac{\pi x}{a_0\xi}\right), \tag{26}$$

where $A(\xi)$ is a dimensionless unknown factor. Substitution into eqn (7) yields the axial force

$$P(t) = \frac{Eh}{1-v^2}\left(\varepsilon_0 - \left(\frac{\pi h A}{2a_0\xi}\right)^2\right). \tag{27}$$

The time-derivatives of the deflection are given by

$$w_{,t}/h = \dot{\xi}\left(A'\left(1-\cos\frac{\pi x}{a}\right) - \frac{\pi x A}{a\xi}\sin\frac{\pi x}{a}\right), \tag{28a}$$

$$w_{,tt}/h = \ddot{\xi}\left(A'\left(1-\cos\frac{\pi x}{a}\right) - \frac{\pi x A}{a\xi}\sin\frac{\pi x}{a}\right)$$
$$+\dot{\xi}^2\left(A''\left(1-\cos\frac{\pi x}{a}\right) + \left(\frac{\pi x}{a\xi}\right)^2 A\cos\frac{\pi x}{a} - 2\left(\frac{A'}{\xi} - \frac{A}{\xi^2}\right)\frac{\pi x}{a}\sin\frac{\pi x}{a}\right), \tag{28b}$$

where the primes indicate differentiation with respect to $\xi$. Multiplying eqn (3) by the right-hand side of eqn (26) and integrating the result over the length of the delamination, one obtains, with the use of eqns (26)–(28), (9), (10) and (23):

$$\left(\frac{3AA'}{2} + \frac{3A^2}{4\xi}\right)(V^2)' + \left(3AA'' + \frac{3}{\xi}AA' - (4\pi^2/3 - 1/2)(A/\xi)^2\right)V^2$$
$$= \left(\frac{\alpha}{\xi^2} - (1+3A^2)\frac{1}{\xi^4}\right)A^2, \tag{29}$$

where

$$V(\xi) = a_0\dot{\xi}/(2\pi c_0). \tag{30}$$

Equation (29) must be supplemented by the energy-balance condition. We first consider a delaminated layer of length $2a_0\xi$ and calculate the reduction of its membrane strain energy as it deforms from a purely membrane state to the final buckled state under the fixed axial strain $\varepsilon_0$. The result is

$$\frac{Eha}{1-v^2}\left[\varepsilon_0^2 - \left(\frac{(1-v^2)P}{Eh}\right)^2\right] = \frac{Eha_0\xi\varepsilon_0^2}{1-v^2}\left(2 - \frac{3A^2}{\alpha\xi^2}\right)\frac{3A^2}{\alpha\xi^2}.$$

For a delaminated layer of fixed length $2a_0$ under the same axial strain, the reduction of the membrane strain energy in the buckling process may be obtained by setting $\xi = 1$ in the last equation. Hence as the delamination length increases from $2a_0$ to $2a_0\xi$, the release of the membrane energy is

$$-\Delta U_m = \frac{3Eha_0\varepsilon_0^2}{1-\nu^2}\left[\frac{A(\xi)^2}{\alpha\xi}\left(2-\frac{3A(\xi)^2}{\alpha\xi^2}\right)-\frac{A(1)^2}{\alpha}\left(2-\frac{3A(1)^2}{\alpha}\right)\right].$$

In the growth process, the increment of the bending and kinetic energies are, respectively,

$$\Delta U_b = \frac{Eh^5}{24(1-\nu^2)a_0^3}\,[\xi(\pi/\xi)^4(A(\xi)/\alpha)^2-\pi^4(A(1)/\alpha)^2]$$

and

$$\Delta T = \frac{Eh^5(\pi V)^2}{24(1-\nu^2)a_0^3}\,\xi\left(3(A')^2+3AA'/\xi+(4\pi^2/3-1/2)\left(\frac{A}{\xi}\right)^2\right).$$

The fracture energy consumed in the growth process is

$$\Delta W = 2(\xi-1)a_0G^* = \frac{(G^*/G_0^*)Eh^5\pi^4}{144(1-\nu^2)a_0^3}(\alpha-1)(\alpha+3)(\xi-1).$$

The energy-balance condition

$$\Delta U_m+\Delta U_b+\Delta T+\Delta W = 0$$

yields the equation

$$[3(A')^2+3AA'/\xi+(4\pi^2/3-1/2)(A/\xi)^2]V^2 = \frac{A^2}{\xi^2}\left(\alpha-\frac{1}{\xi^2}-\frac{3A^2}{2\xi^2}\right)-\frac{1}{6\xi}(\alpha-1)^2-\left(\frac{G^*}{G_0^*}\right)\frac{(\alpha-1)(\alpha+3)}{6}(1-1/\xi). \quad (31)$$

Equations (29) and (31) form a coupled system of nonlinear differential equations for the dimensionless amplitude $A(\xi)$ and the normalized crack speed $V(\xi)$. They must be supplemented by three initial conditions, two of which are evident:

$$A(1) = [(\alpha-1)/3]^{1/2},\quad V(1) = 0. \quad (32a, b)$$

A third condition may be obtained by requiring that eqns (29) and (31) deliver the same initial value for $(V^2)''$. Differentiating the first equation once and the second equation twice with respect to $\xi$, and setting $\xi = 1$, one obtains

$$(\tfrac{1}{2}\sqrt{3/(\alpha-1)}A'+\tfrac{1}{4})(V^2)'' = -2\sqrt{(\alpha-1)/3}A'+2\alpha/3$$

and

$$\left(3(A')^2+\sqrt{3(\alpha-1)}A'+\left(\frac{4\pi^2}{3}-\frac{1}{2}\right)\frac{\alpha-1}{3}\right)(V^2)'' = -4(\alpha-1)(A')^2+8\alpha\sqrt{(\alpha-1)/3}A'-\tfrac{4}{3}(\alpha-1)(\alpha+3/2).$$

These relations yield identical values of $(V^2)''$ if

$$(3\alpha-1)(A')^2+2\sqrt{(\alpha-1)/3}\left(\left(\frac{4\pi^2}{3}-\frac{1}{2}\right)\frac{\alpha-1}{3}-\alpha-\frac{3}{2}\right)A'$$
$$-\frac{\alpha-1}{3}\left(\alpha+\frac{3}{2}+\frac{2}{3}\left(\frac{4\pi^2}{3}-\frac{1}{2}\right)\alpha\right)=0. \quad (33)$$

The positive root of this quadratic equation gives the initial value $A'(1)$.

To facilitate numerical integration, the system of equations (29) and (31) and the associated initial conditions (32) and (33) will be transformed into an equivalent initial-value problem involving two first-order equations. We let

$$Y(\xi)=\xi[A(\xi)]^2, \quad W(\xi)=Y'(\xi), \quad (34a, b)$$

$$Q(\xi)=[W(\xi)V(\xi)]^2, \quad (34c)$$

$$f(\xi,Y)=\frac{Y}{\xi^2}\left(\alpha-\frac{1}{\xi^2}-\frac{3Y}{2\xi^3}\right)+(\alpha+1)\left(\frac{2}{3}-\xi\left(\frac{\alpha}{6}+\frac{1}{2}\right)\right). \quad (34d)$$

Then eqns (31) and (29) become, respectively

$$\left(\frac{3W^2}{4Y}+\left(\frac{4\pi^2}{3}-\frac{5}{4}\right)\frac{Y}{\xi^2}\right)V^2=f(\xi,Y), \quad (35)$$

$$Q'=\frac{4W}{3}\left(f(\xi,Y)+Y\left[\alpha/\xi^2-\frac{1}{\xi^2}(1+3Y/\xi)\right]\right). \quad (36)$$

Equations (34b) and (36) now yield

$$\frac{dQ}{dY}=\frac{4}{3}\left(f(\xi,Y)+Y\left[\alpha/\xi^2-\frac{1}{\xi^2}(1+3Y/\xi)\right]\right), \quad (37a)$$

$$\frac{d\xi}{dY}=\frac{\xi}{Y}\left(\frac{f(\xi,Y)Y/Q-3/4}{4\pi^2/3-5/4}\right)^{1/2}. \quad (37b)$$

In this new system of equations, $Y=\xi A^2$ replaces $\xi$ as the independent variable. The associated initial conditions are

$$Q=0 \quad \text{and} \quad \xi=1 \quad \text{at} \quad Y=(\alpha-1)/3. \quad (38)$$

In order to start the numerical integration procedure for (37) and (38), one must know the initial value of $d\xi/dY$ (since the right-hand side of eqn (37b) is indeterminate at the initial point). This is given by

$$\frac{1}{Y'(1)}=\frac{1}{2\sqrt{(\alpha-1)/3}A'(1)+(\alpha-1)/3},$$

where $A'(1)$ is determined by eqn (33).

For various values of $\alpha$, eqns (37) and (38) are integrated numerically by using the fourth-order Runge–Kutta formula. The solutions $Q(Y)$ and $\xi(Y)$ determine the normalized growth speed $V$ according to eqn (34c), i.e.

$$V=\xi\left(\frac{f(\xi,Y)Q/Y-\frac{3}{4}(Q/Y)^2}{4\pi^2/3-5/4}\right)^{1/2}. \quad (39)$$

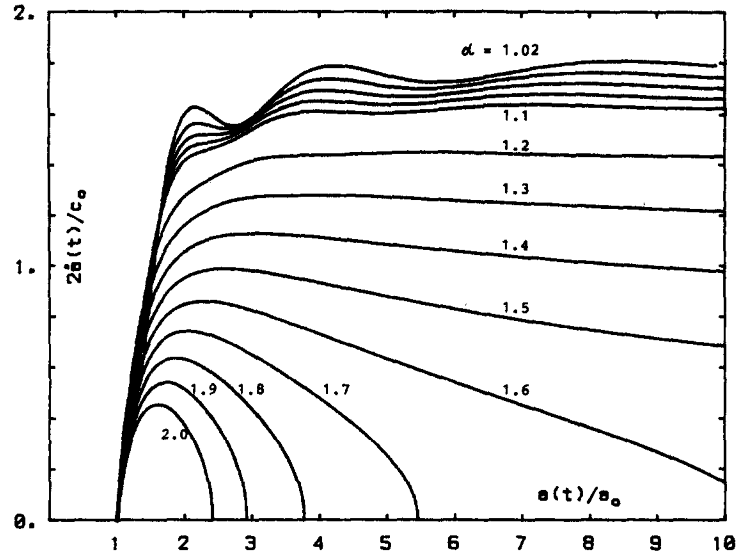


Fig. 3. Uni-directional growth speed—refined solution.

The results are shown in Fig. 3. Comparison of Figs 2 and 3 shows excellent agreement between the quasi-dynamic solution and the improved solution for $\alpha > 1.5$, i.e. if the specific fracture energy is large enough for the delamination growth to be eventually arrested. The discrepancies between the growth speeds of the two models become significant as $\alpha$ decreases, and the wave speed of the improved solution may fluctuate over time when $\alpha$ is close to unity. Further comparison between the two solutions is shown in Fig. 4, where the ratio of their deflection amplitudes, $\{3/(\alpha-1)\}^{1/2}A(\xi)$, is plotted against the delamination length. The ratio is uniformly close to unity for $\alpha > 1.5$. When $\alpha$ is close to unity, the deflection amplitude of the improved solution may be significantly smaller than the amplitude of the quasi-dynamic solution in an initial stage of delamination growth.

## 5. BI-DIRECTIONAL GROWTH

If the delamination grows with the same speed at both ends, then the mid-point of the delamination remains fixed and it is convenient to define the point as the origin of the $x$-

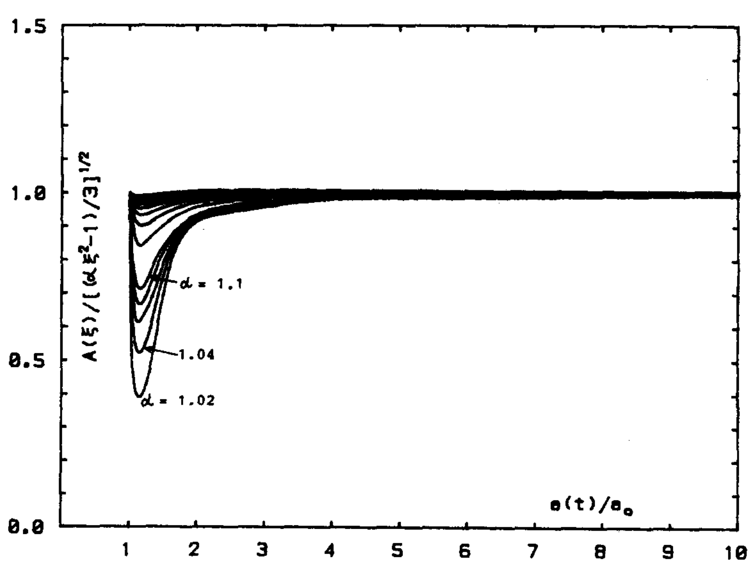


Fig. 4. Amplitude ratio of the refined vs the quasidynamic solution—uni-directional growth.

coordinate. For this case, the quasi-dynamic analysis and the improved analysis proceed in ways similar to the case of uni-directional growth. The only change in the result is that the constant $4\pi^2/3$ in eqns (22), (24), (25), (29), (31), (33), (35), (37b) and (39) is replaced by $\pi^2/3$. The normalized delamination growth speeds calculated from the two solutions are shown in Figs 5 and 6. It should be mentioned that, whereas in uni-directional growth only one crack tip moves with the speed $2\dot{a}(t)$, in bi-directional growth both crack tips move with the speed $a(t)$. The figures indicate that, compared to uni-directional delamination growth, bi-directional growth shows a smaller speed of crack-tip movement but a greater rate of increase of the delamination length. The amplitude ratio of the refined solution versus the quasi-dynamic solution is shown in Fig. 7 for the case of bi-directional growth.

## 6. CONCLUDING REMARKS

The preceding results show that, if the strain load in the base laminate, $\varepsilon_0$, is maintained constant after the initiation of delamination growth, then the nature of dynamic growth

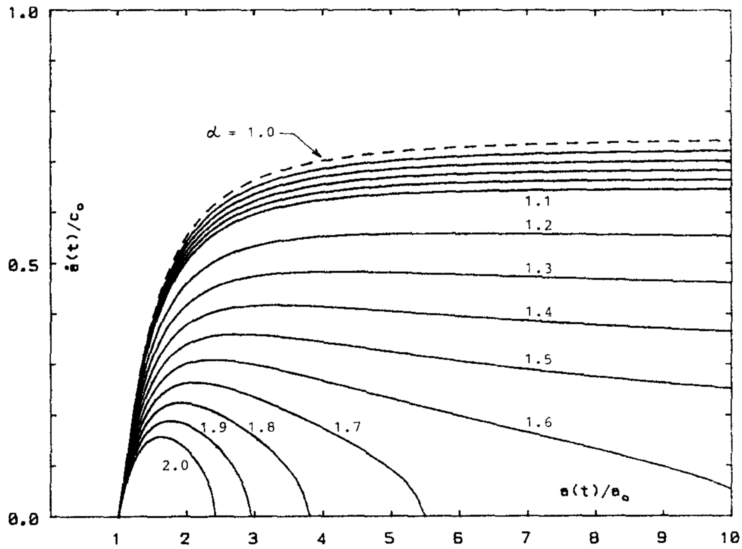


Fig. 5. Bi-directional growth speed—quasidynamic solution.

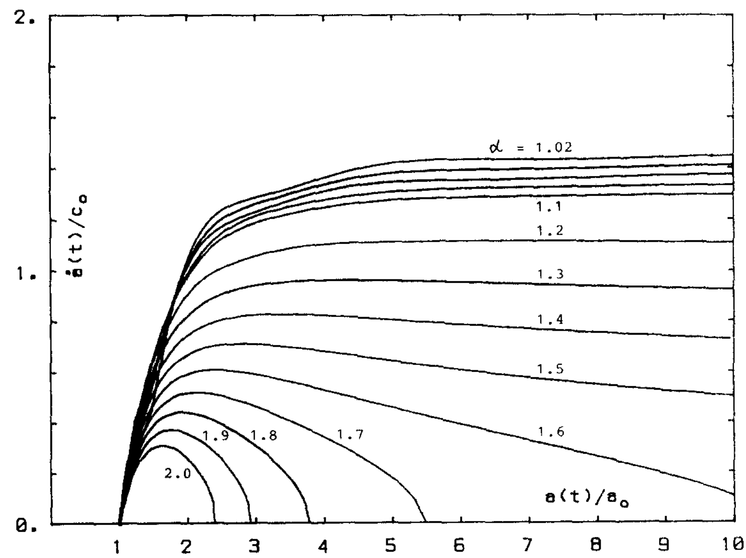


Fig. 6. Bi-directional growth speed—refined solution.

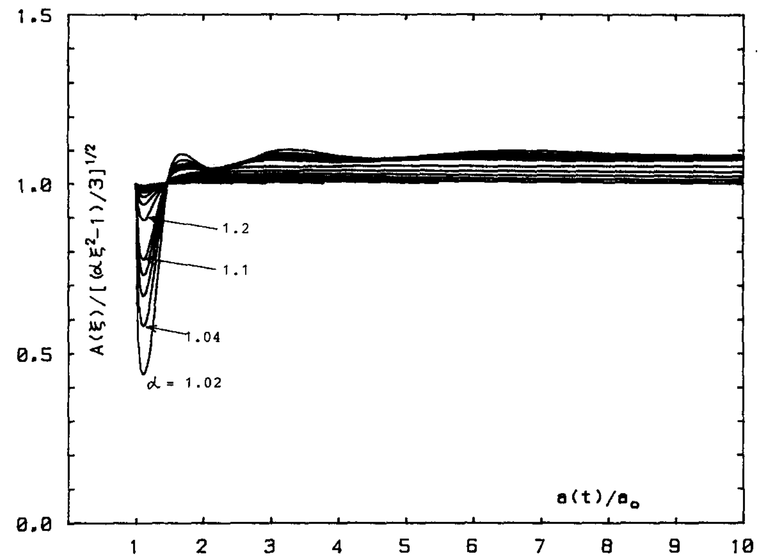


Fig. 7. Amplitude ratio of the refined vs the quasidynamic solution—bi-directional growth.

depends on the nondimensionalized parameter $\gamma^*$ or $\alpha$ [see eqns (9) and (10)]. $\gamma^*$ and $\alpha$ are not purely material parameters because they depend on the initial geometry of the delaminated layer through the factor $a_0^4 h^5$ in eqn (9a). For a relatively large $\alpha$ (i.e. $1.5 < \alpha < 3$), delamination growth at first accelerates and subsequently decelerates to a state of arrest. The maximum growth speed was found to be smaller than the flexural wave speed $c_0$ (the speed at which the initial buckling form propagates along the delaminated layer). The quasi-dynamic analysis and the refined analysis yield exceedingly close estimates of the growth speed and the amplitude of deflection. Experimental observation has confirmed that the crack growth speed for a thin surface delamination (i.e. with a relatively large $a_0/h$) is generally comparable to the flexural wave speed in the delaminated layer (Takeda *et al.*, 1982).

On the other hand, delamination growth under the fixed strain load $\varepsilon_0$ continues without arrest if the parameter $\alpha$ is smaller than 1.5. Crack growth at first accelerates and subsequently the growth speed approaches a limiting value comparable to the flexural wave speed [eqn (25)]. The refined analysis generally yields smaller growth speeds compared to the quasi-dynamic analysis. The differences are small in the case of bi-directional growth. In uni-directional growth (which has a relatively larger crack growth speed) the discrepancies between the two solutions become significant as $\alpha$ approaches unity. Furthermore, the crack speed of the refined solution fluctuates in the growth process.

Both solutions violate the dynamic growth condition at the crack tip [eqn (8)]. In fast delamination growth the energy-release rates calculated from the two approximate solutions may be several times greater than the specific fractue energy $G_0^*$. This suggests that the deflection function determined by the quasi-dynamic or the improved analysis is approximately valid only in an interior segment of the delaminated layer. In a boundary segment adjacent to the moving crack tip, the true curvature of the layer is significantly curtailed in compliance with the local growth condition [eqn (8)]. The curvature may even vanish and delamination growth may be sustained only by shear action between the layer and the base plate that are in partial contact.

The dynamic delamination growth problem, as defined by eqns (3)–(8), has been solved numerically by Chen and Yin (1988) using the finite-difference method in the space-time domain, without any *a priori* assumption of the shape of the deflection function. A comparison of the growth speeds of the finite-difference dynamic solution with the approximate solution of the present analysis is shown in Fig. 8 [reproduced from Chen and Yin (1988)]. The two sets of results, indicated respectively by fluctuating solid curves and smooth dashed curves, show close agreement in those cases where delamination growth eventually

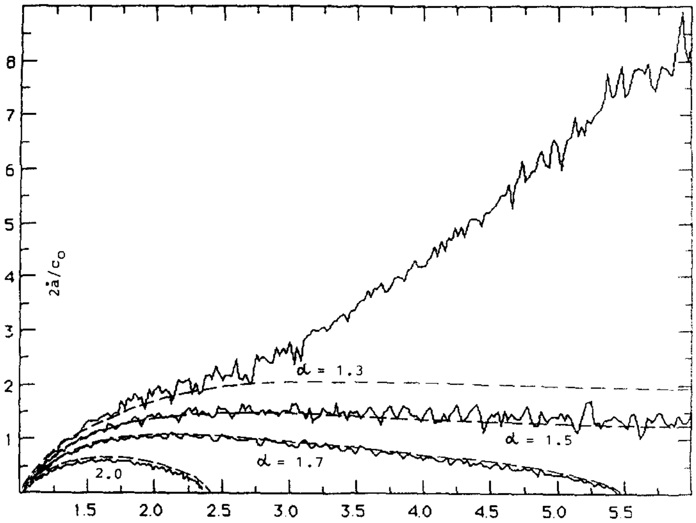


Fig. 8. Comparison of the growth speeds—quasi-dynamic solution vs the finite-difference solution of the dynamic problem.

terminated in a state of arrest (see the three pairs of curves with $\alpha \geqslant 1.5$ in Fig. 8). In the contrary case of accelerating, catastrophic growth ($1 < \alpha < 1.5$), the dynamic solution yields a crack growth speed which is initially close to the present solution but becomes much larger subsequently.

The deflection functions of the dynamic solution have been computed recently by Chen and Ngo (1989), and compared with the static deflections. If $\alpha > 1.5$, the dynamic deflection remains in close agreement with the static deflection during the growth process. For $\alpha < 1.5$, the dynamic deflection becomes increasingly smaller than the static deflection as growth continues, and much of the difference is due to the significantly reduced curvature and slope of the dynamic deflection near the crack tip [see Figs 10–12 in Chen and Ngo (1989)].

If crack growth accelerates to a speed significantly greater than the speeds of flexural waves in the delaminated layer, then the evolution of the bending deformation of the buckled layer cannot reach and *directly* affect the crack-tip region, which is moving ahead faster than the flexural waves. Therefore, the instantaneous deflection of the delaminated layer may depart increasingly from the equilibrium deflection assumed in the quasi-dynamic analysis. However, the evolution of dynamic deflection can *indirectly* affect the crack-tip stress field through the axial decompression of the layer, which is due to the nonliner term $(w')^2/2$ in eqn (7), because the decompression propagates as longitudinal waves which have a much greater wave speed depending only on the material and not on the length or thickness of the layer. The decompression causes an abrupt change in the axial stress across a short distance in the crack-tip region above the plane of delamination, while no such drastic change of the axial stress exists in the region below the plane of delamination. As a result, the moving crack-tip region is subjected to a severe shear action. Thus, catastrophic delamination growth may begin as a predominantly model I fracture action in which the crack-tip stress field is essentially affected by flexural waves in the delaminated layer, and subsequently develop, with an increasing crack speed, into predominantly mode II fracture behavior driven by axial decompression of the layer and intense shear action in the crack tip region.

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